

The second largest number of points of plane curves over finite fields

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Abstract

A basis of the ideal of the complement of a linear subspace in a projective space over a finite field is given. As an application, the second largest number of points of plane curves of degree d over the finite field of q elements is also given for $d \geq q + 1$.

Key Words: Finite field, Basis of the ideal, Plane curve

MSC: 11G20, 13F20, 14G15, 14N05

1 Introduction

Let C be a curve of degree d in projective plane \mathbb{P}^2 defined over the finite field \mathbb{F}_q of q elements, which has no \mathbb{F}_q -line components, and $F(X, Y, Z)$ a homogeneous polynomial in $\mathbb{F}_q[X, Y, Z]$ which defines the curve C . We are interesting in the set

$$C(\mathbb{F}_q) := \{(a, b, c) \in \mathbb{P}^2(\mathbb{F}_q) \mid F(a, b, c) = 0\},$$

where $\mathbb{P}^2(\mathbb{F}_q)$ denotes the set of \mathbb{F}_q -points of \mathbb{P}^2 . The number of $C(\mathbb{F}_q)$ is denoted by $N_q(C)$.

In the series of papers [1, 2, 3], we proved the Sziklai bound, which says that

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$N_q(C) \leq (d-1)q + 1$ unless C is a curve over \mathbb{F}_4 which is projectively equivalent to the curve defined by

$$(X + Y + Z)^4 + (XY + YZ + ZX)^2 + XYZ(X + Y + Z) = 0$$

over \mathbb{F}_4 .

In order to give a brief explanation of what we will do, we should explain some notation.

Notation 1.1 The number of points of $\mathbb{P}^n(\mathbb{F}_q)$ is frequently denoted by $\theta_q(n)$, that is $\theta_q(n) = \frac{q^{n+1}-1}{q-1}$.

Let x_0, \dots, x_n be coordinates of \mathbb{P}^n , and f_1, \dots, f_n homogeneous polynomials over \mathbb{F}_q . The algebraic set in \mathbb{P}^n over the algebraic closure of \mathbb{F}_q defined by $f_1 = \dots = f_n = 0$ is frequently denoted by $\{f_1 = \dots = f_n = 0\}$.

Here we summarize symbols related to plane curves, which will be used in this paper or future, and also agree with ones in [4].

- $\mathcal{C}_d(\mathbb{F}_q)$: the set of plane curves of degree d over \mathbb{F}_q without \mathbb{F}_q -linear components.
- $\mathcal{C}_d^i(\mathbb{F}_q) := \{C \in \mathcal{C}_d(\mathbb{F}_q) \mid C \text{ is absolutely irreducible}\}$
- $\mathcal{C}_d^s(\mathbb{F}_q) := \{C \in \mathcal{C}_d(\mathbb{F}_q) \mid C \text{ is nonsingular}\}$
- $M_q(d) = \max\{N_q(C) \mid C \in \mathcal{C}_d(\mathbb{F}_q)\}$
- $M_q^i(d) = \max\{N_q(C) \mid C \in \mathcal{C}_d^i(\mathbb{F}_q)\}$
- $M_q^s(d) = \max\{N_q(C) \mid C \in \mathcal{C}_d^s(\mathbb{F}_q)\}$
- ${}_2M_q(d) = \max\{N_q(C) \mid C \in \mathcal{C}_d(\mathbb{F}_q), N_q(C) < M_q(d)\}$
- ${}_2M_q^i(d) = \max\{N_q(C) \mid C \in \mathcal{C}_d^i(\mathbb{F}_q), N_q(C) < M_q^i(d)\}$
- ${}_2M_q^s(d) = \max\{N_q(C) \mid C \in \mathcal{C}_d^s(\mathbb{F}_q), N_q(C) < M_q^s(d)\}$

Since the number of $\mathbb{P}^2(\mathbb{F}_q)$ is $\theta_q(2)$, the Sziklai bound makes sense in the range $2 \leq d \leq q+2$. We should add a few words to the Sziklai bound: for $d = 2$, $\sqrt{q} + 1$ (if q is square), $q-1$, q , $q+1$ and $q+2$, this bound is sharp. For those degrees d , it might be interesting to know the second largest number ${}_2M_q(d)$. In this paper, we try to find it for $d = q+1$ and $q+2$. Furthermore, since we have already knew $M_q(d) = \theta_q(2)$ if $d \geq q+3$ [1, Prop. 1.1], we can determine ${}_2M_q(d)$ for $d \geq q+3$.

More precisely, we will show the following facts.

Theorem 1.2 (1) *Suppose that $q > 3$.*

- (i) *If $d \geq 2q-1$, then ${}_2M_q(d) = \theta_q(2) - 1 = q^2 + q$.*

- (ii) If $2q - 1 \geq d \geq q + 2$, then ${}_2M_q(d) = \theta_q(2) - (2q - d) = q^2 + d - q + 1$.
- (iii) If $d = q + 2$, then ${}_2M_q(q + 2) = {}_2M_q^i(q + 2) = q^2 + 3$.
- (2) For any q , we have ${}_2M_q(q + 1) = {}_2M_q^i(q + 1) = q^2$.

Figure 1 illustrates the main theorem. In the picture \bullet indicates existence of a plane curve of assigned degree and number of points, and \times non-existence.

The important ingredient of the proof is to determine a basis of the ideal of $\mathbb{P}^2(\mathbb{F}_q) \setminus \{\text{an } \mathbb{F}_q\text{-point}\}$ for (1) of Theorem 1.2, and $\mathbb{P}^2(\mathbb{F}_q) \setminus \{\text{an } \mathbb{F}_q\text{-line}\}$ for (2). We formulate them under a little more general setting in Theorem 2.2 below.

2 Ideal of the complement of a linear subspace in $\mathbb{P}^n(\mathbb{F}_q)$

In this section, we consider two subsets of \mathbb{F}_q -points of \mathbb{P}^n with homogeneous coordinates x_0, x_1, \dots, x_n .

The following lemma, which is easy to show, is a kind of folklore.

Lemma 2.1 *The ideal of $\mathbb{P}^n(\mathbb{F}_q)$ in $\mathbb{F}_q[x_0, \dots, x_n]$ is generated by $\{x_i^q x_j - x_i x_j^q \mid 0 \leq i < j \leq n\}$.*

Theorem 2.2 *Let k be an integer with $1 \leq k \leq n$, and \mathbb{P}^{k-1} the linear subspace defined by $x_k = x_{k+1} = \dots = x_n = 0$. Then the ideal of $\mathbb{P}^n(\mathbb{F}_q) \setminus \mathbb{P}^{k-1}(\mathbb{F}_q)$ in $\mathbb{F}_q[x_0, \dots, x_n]$ is generated by*

$$\{x_i^q x_j - x_i x_j^q \mid 0 \leq i < j \leq n\} \cup \{x_s \prod_{i=k}^n (x_i^{q-1} - x_s^{q-1}) \mid s = 0, 1, \dots, k-1\}.$$

Proof. (Step 1) In the first step, we handle the case where the linear subspace is of dimension 0, that is, $k = 1$.

Let J_n be the ideal of $\mathbb{P}^n(\mathbb{F}_q) \setminus \{(1, 0, \dots, 0)\}$ in $\mathbb{F}_q[x_0, \dots, x_n]$, and I_n the ideal generated by

$$\{x_i^q x_j - x_i x_j^q \mid 0 \leq i < j \leq n\} \cup \{x_0 \prod_{i=1}^n (x_i^{q-1} - x_0^{q-1})\}.$$

It is obvious that $J_n \supset I_n$. We will show $J_n = I_n$ by induction on n . For $n = 1$,

$$I_1 = (x_0^q x_1 - x_0 x_1^q, x_0(x_1^{q-1} - x_0^{q-1})) = (x_0(x_1^{q-1} - x_0^{q-1}))$$

is trivially the ideal of $\{(\lambda, 1) \mid \lambda \in \mathbb{F}_q\} = \mathbb{P}^1(\mathbb{F}_q) \setminus \{(1, 0)\}$.

Next we assume that $n \geq 2$. Let $F(x_0, \dots, x_n)$ be a homogeneous polynomial of degree d in J_n . We want to show that $F(x_0, \dots, x_n) \equiv 0 \pmod{I_n}$. First we prove the following claim:

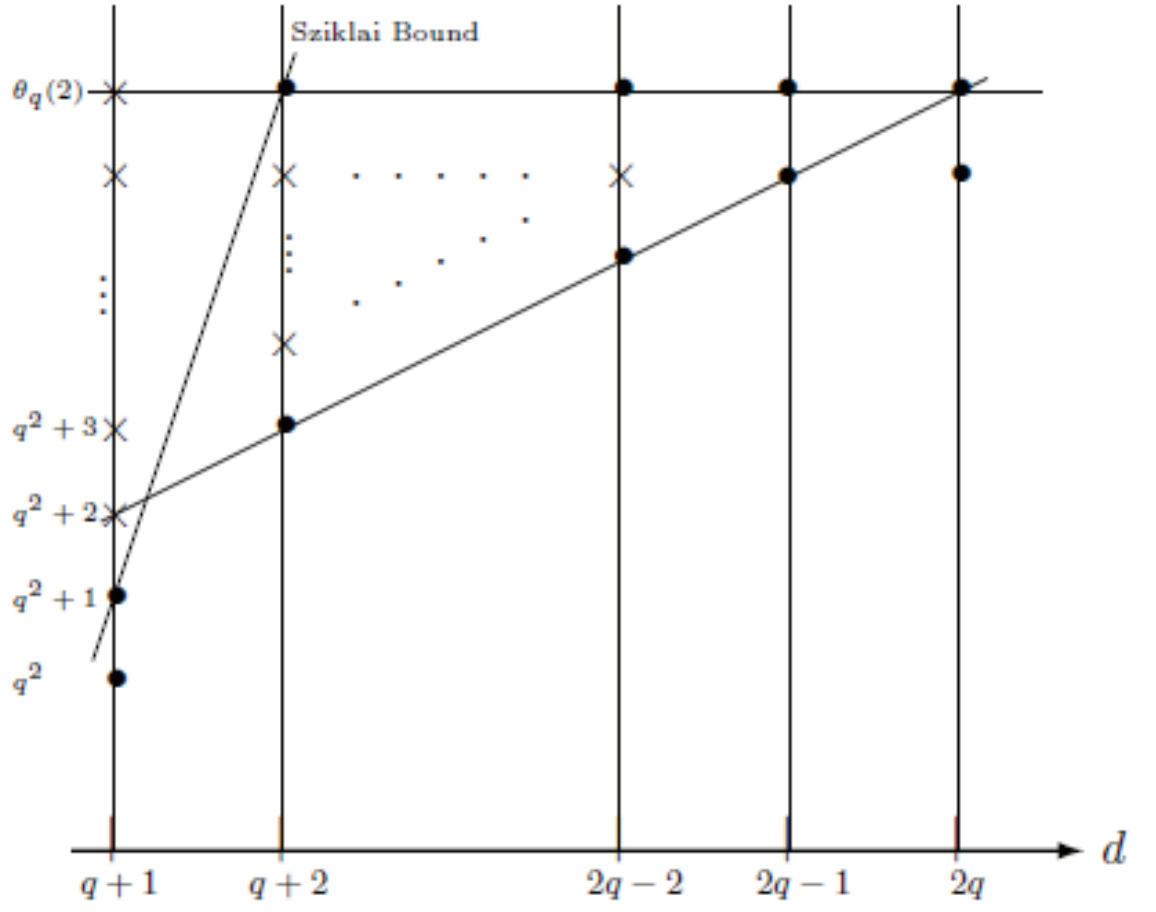


Figure 1: $M_q(d)$ and ${}_2M_q(d)$ for $d \geq q+1$ with $q > 3$

Claim. We have

$$F(x_0, \dots, x_n) \equiv \sum_{i=0}^{q-1} f_i(x_0, \dots, x_{n-1}) x_n^i \pmod{I_n},$$

where $f_i(x_0, \dots, x_{n-1})$ is homogeneous of degree $d - i$.

When $d \leq q - 1$, there is nothing to do. Suppose $d \geq q$. For a monomial $M = x_0^{e_0} \cdots x_n^{e_n}$ which appears in F , if $q \leq e_n < d$, then $M \equiv x_0^{e'_0} \cdots x_{n-1}^{e'_{n-1}} x_n^{e'_n} \pmod{I_n}$ with $e'_n < e_n$, because $x_i^q x_n - x_i x_n^q \in I_n$. (Note that since $e_n < d$, there is an e_i with $0 \leq i \leq n - 1$ such that $e_i \geq 1$.) Repeating this procedure, we get $M \equiv x_0^{e'_0} \cdots x_{n-1}^{e'_{n-1}} x_n^{e'_n} \pmod{I_n}$ with $e'_n < q$. Hence

$$F(x_0, \dots, x_n) \equiv c x_n^d + \sum_{i=0}^{q-1} f_i(x_0, \dots, x_{n-1}) x_n^i \pmod{I_n}. \quad (1)$$

Since the right-hand side of (1) is also an element of J_n , $(0, \dots, 0, 1)$ is its zero. Hence $c = 0$.

From this claim, we may assume that

$$F(x_0, \dots, x_n) = \sum_{i=0}^{q-1} f_i(x_0, \dots, x_{n-1}) x_n^i, \quad \deg f_i = d - i.$$

Let $(\lambda_0, \dots, \lambda_{n-1}) \in \mathbb{F}_q^n \setminus \{(1, 0, \dots, 0)\}$. Since $F(\lambda_0, \dots, \lambda_{n-1}, \mu) = 0$ for any $\mu \in \mathbb{F}_q$ and $F(\lambda_0, \dots, \lambda_{n-1}, x_n)$ is of degree $q - 1$ in x_n , $f_\nu(\lambda_0, \dots, \lambda_{n-1}) = 0$ for $\nu = 0, \dots, q - 1$. Therefore $f_\nu(x_0, \dots, x_{n-1}) \in I_{n-1}$ by the induction hypothesis, that is,

$$f_\nu(x_0, \dots, x_{n-1}) = \sum_{0 \leq i < j \leq n-1} h_{ij} \cdot (x_i^q x_j - x_i x_j^q) + k_\nu x_0 \prod_{i=1}^{n-1} (x_i^{q-1} - x_0^{q-1}),$$

where $k_\nu = k_\nu(x_0, \dots, x_{n-1})$ is of degree $d - \nu - ((n - 1)(q - 1) + 1)$, say d_ν . (Note that if $d - \nu - ((n - 1)(q - 1) + 1) < 0$, we understand $k_\nu = 0$.) Put

$$k_\nu(x_0, \dots, x_{n-1}) = c_\nu x_0^{d_\nu} + k_\nu^l(x_0, \dots, x_{n-1}),$$

where the degree in x_0 of k_ν^l is less than d_ν . Then

$$F(x_0, \dots, x_n) \equiv \sum_{\nu=0}^{q-1} (c_\nu x_0^{d_\nu} + k_\nu^l(x_0, \dots, x_{n-1})) x_0 \prod_{i=1}^{n-1} (x_i^{q-1} - x_0^{q-1}) x_n^\nu \pmod{I_n}.$$

So we may assume that

$$F(x_0, \dots, x_n) = G(x_0, \dots, x_n) + H(x_0, \dots, x_n)$$

with

$$G(x_0, \dots, x_n) = x_0 \prod_{i=1}^{n-1} (x_i^{q-1} - x_0^{q-1}) \left(\sum_{\nu=0}^{q-1} c_\nu x_0^{d_\nu} x_n^\nu \right)$$

and

$$H(x_0, \dots, x_n) = \sum_{\nu=0}^{q-1} k_\nu^l(x_0, \dots, x_{n-1}) x_0 \prod_{i=1}^{n-1} (x_i^{q-1} - x_0^{q-1}) x_n^\nu.$$

Since $k_\nu^l(x_0, \dots, x_{n-1})$ is the sum of lower terms in x_0 of the homogeneous polynomial $k_\nu(x_0, \dots, x_{n-1})$, each term of k_ν^l contains some x_j with $0 < j \leq n-1$. Since

$$x_j x_0 \prod_{i=1}^{n-1} (x_i^{q-1} - x_0^{q-1}) = \left(\prod_{\substack{i=1 \\ i \neq j}}^{n-1} (x_i^{q-1} - x_0^{q-1}) \right) (x_j^q x_0 - x_j x_0^q) \in I_n,$$

we have $H(x_0, \dots, x_n) \in I_n$.

Finally we show that $G(x_0, \dots, x_n) \in I_n$. For $(\mu, \rho) \in \mathbb{F}_q^2 \setminus \{(1, 0)\}$, $G(\mu, 0, \dots, 0, \rho) = 0$ because $G = F - H \in J_n$. Hence $G(x_0, 0, \dots, 0, x_n) \in I_1$ with respect to coordinates x_0, x_n by the induction hypothesis, that is $G(x_0, 0, \dots, 0, x_n) \in (x_0(x_n^{q-1} - x_0^{q-1}))$. On the other hand, since

$$G(x_0, 0, \dots, 0, x_n) = (-1)^{n-1} x_0^{(n-1)(q-1)+1} \left(\sum_{\nu=0}^{q-1} c_\nu x_0^{d_\nu} x_n^\nu \right),$$

there is a polynomial $u(x_0, x_n)$ such that

$$\sum_{\nu=0}^{q-1} c_\nu x_0^{d_\nu} x_n^\nu = u(x_0, x_n) (x_n^{q-1} - x_0^{q-1}).$$

Therefore $G(x_0, \dots, x_n) = u(x_0, x_n) x_0 \prod_{i=1}^n (x_i^{q-1} - x_0^{q-1}) \in I_n$.

(Step 2) We complete the proof by induction on the dimension of the linear subspace \mathbb{P}^{k-1} . Suppose $k \geq 2$. Let $J_{n,k}$ be the ideal of $\mathbb{P}^n(\mathbb{F}_q) \setminus \mathbb{P}^{k-1}(\mathbb{F}_q)$, and $I_{n,k}$ the ideal generated by $\{x_i^q x_j - x_i x_j^q \mid 0 \leq i < j \leq n\} \cup \{x_s \prod_{i=k}^n (x_i^{q-1} - x_s^{q-1}) \mid s = 0, 1, \dots, k-1\}$. Note that the second subscript in notation of those ideals indicates the dimension of the vector space $H^0(\mathbb{P}^{k-1}, \mathcal{O}(1))$. Obviously $J_{n,k} \supset I_{n,k}$. Let $F(x_0, \dots, x_n)$ be a homogeneous polynomial in $J_{n,k}$. Divide the terms of F into two parts: let $G_1(x_0, x_2, \dots, x_n)$ be the sum of terms of F each of which does not contains x_1 , and $G_2(x_0, x_1, \dots, x_n)$ the sum of terms of F each of which contains x_1 . In other words,

$$\begin{aligned} G_1(x_0, x_2, \dots, x_n) &= F(x_0, 0, x_2, \dots, x_n) \\ G_2(x_0, x_1, \dots, x_n) &= F(x_0, x_1, x_2, \dots, x_n) - F(x_0, 0, x_2, \dots, x_n). \end{aligned} \tag{2}$$

From (2), $G_1(\lambda_0, \lambda_2, \dots, \lambda_n) = 0$ for any $(\lambda_0, \lambda_2, \dots, \lambda_n) \in \mathbb{P}^{n-1} \setminus \{x_k = \dots = x_n = 0\}$ with respect to the coordinates $x_0, \hat{x}_1, x_2, \dots, x_{k-1}; x_k, \dots, x_n$, where \hat{x}_1 means omitting x_1 . Hence, by induction hypothesis¹, $G_1(x_0, x_2, \dots, x_n) \in I_{n-1, k-1}$ with

¹ The dimension of the linear subspace $\{x_k = \dots = x_n = 0\}$ in \mathbb{P}^{n-1} with respect to the coordinates $x_0, \hat{x}_1, x_2, \dots, x_n$ is $k-2$.

respect to the coordinates $x_0, \hat{x}_1, x_2, \dots, x_{k-1}; x_k, \dots, x_n$, that is, G_1 is contained in the ideal generated by

$$\{x_i^q x_j - x_i x_j^q \mid 0 \leq i < j \leq n, i, j \neq 1\} \cup \{x_s \prod_{i=k}^n (x_i^{q-1} - x_s^{q-1}) \mid s = 0, 2, \dots, k-1\}.$$

Hence G_1 is an element of the original $I_{n,k}$, and hence $G_2 = F - G_1 \in J_{n,k}$. Since $x_0^q x_1 - x_0 x_1^q \in I_{n,k}$ and each term of $G_2(x_0, x_1, \dots, x_n)$ contains x_1 ,

$$G_2(x_0, x_1, \dots, x_n) \equiv \sum_{i=0}^{q-1} g_i(x_1, \dots, x_n) x_0^i \pmod{I_{n,k}}. \quad (3)$$

Hence it is enough to observe that the right-hand side of (3) belongs $I_{n,k}$. Since the polynomial already contained in $J_{n,k}$, for a fixed $(\lambda_1, \dots, \lambda_n) \in \mathbb{F}_q^n \setminus \{\lambda_k = \dots = \lambda_n = 0\}$ and an arbitrary $\mu \in \mathbb{F}_q$,

$$\sum_{i=0}^{q-1} g_i(\lambda_1, \dots, \lambda_n) \mu^i = 0.$$

Since $\sum_{i=0}^{q-1} g_i(\lambda_1, \dots, \lambda_n) x_0^i$ is a polynomial of degree $q-1$ in x_0 , we have $g_i(\lambda_1, \dots, \lambda_n) = 0$ for each $i = 0, 1, \dots, q-1$. Hence $g_i(x_1, \dots, x_n) \in I_{n-1,k-1}$ with respect to the variables $x_1, \dots, x_{k-1}; x_k, \dots, x_n$ by induction hypothesis. It is obvious that this ideal $I_{n-1,k-1}$ is contained in $I_{n,k}$. \square

The following corollary is just the case $k = 1$ in Theorem 2.2, which has been proved in Step 1.

Corollary 2.3 *The ideal of $\mathbb{P}^n(\mathbb{F}_q) \setminus \{(1, 0, \dots, 0)\}$ in $\mathbb{F}_q[x_0, \dots, x_n]$ is generated by*

$$\{x_i^q x_j - x_i x_j^q \mid 0 \leq i < j \leq n\} \cup \{x_0 \prod_{i=1}^n (x_i^{q-1} - x_0^{q-1})\}.$$

Corollary 2.4 *Let $P_0 \in \mathbb{P}^n(\mathbb{F}_q)$. Then there is a homogeneous polynomial $F(x_0, \dots, x_n)$ of degree d in $\mathbb{F}_q[x_0, \dots, x_n]$ such that the hypersurface H defined by $F = 0$ satisfies $H(\mathbb{F}_q) = \mathbb{P}^n(\mathbb{F}_q) \setminus \{P_0\}$ if and only if $d \geq (q-1)n + 1$.*

Proof. It is obvious that we may choose $P_0 = (1, 0, \dots, 0)$. Since polynomials $x_i^q x_j - x_i x_j^q$ ($0 \leq i < j \leq n$) vanish on the entire $\mathbb{P}^n(\mathbb{F}_q)$, the “only if” part follows from the above corollary immediately. Conversely the hypersurface H defined by $x_0^{d-(q-1)n} \prod_{i=1}^n (x_i^{q-1} - x_0^{q-1})$ has the desired property if $d > (q-1)n$. \square

Corollary 2.5 *Let \mathbb{A}^n be the affine part of \mathbb{P}^n defined by $x_n \neq 0$. The ideal of $\mathbb{A}^n(\mathbb{F}_q)$ in $\mathbb{F}_q[x_0, \dots, x_n]$ is generated by*

$$\{x_s x_n^{q-1} - x_s^q \mid 0 \leq s < n\}.$$

Proof. From Theorem 2.2, the ideal is generated by

$$\{x_i^q x_j - x_i x_j^q \mid 0 \leq i < j \leq n\} \cup \{x_s x_n^{q-1} - x_s^q \mid 0 \leq s < n\}.$$

Since

$$x_i^q x_j - x_i x_j^q = (x_j x_n^{q-1} - x_j^q) x_i - (x_i x_n^{q-1} - x_i^q) x_j,$$

the ideal is generated by only the latter set. \square

3 The second largest number for $d \geq q + 1$

Throughout this section we fix projective plane \mathbb{P}^2 with coordinates X, Y, Z over \mathbb{F}_q .

The following lemma is a corollary of the Sziklai bound, which guarantees a plane curve without \mathbb{F}_q -line components having many \mathbb{F}_q -points to be absolutely irreducible.

Lemma 3.1 *Let C be a curve over \mathbb{F}_q of degree d in \mathbb{P}^2 without \mathbb{F}_q -linear components. If $N_q(C) \geq (d - 2)q + 3$, then C is absolutely irreducible.*

Proof. See [5, Corollary 2.2]. \square

3.1 The case $q + 2 \leq d \leq 2q - 1$

Lemma 3.2 *Let C be a curve over \mathbb{F}_q of degree d in \mathbb{P}^2 with $q + 2 \leq d \leq 2q - 2$, which may have \mathbb{F}_q -linear components. Suppose $C(\mathbb{F}_q) \neq \mathbb{P}^2(\mathbb{F}_q)$. Let $N_q(C) = \theta_q(2) - (m + 1)$. Then $m + 1 \geq 2q - d$, that is $N_q(C) \leq q^2 + (d - q + 1)$; and if equality occurs, then those $m + 1$ missing points are collinear.*

Proof. Let $F \in \mathbb{F}_q[X, Y, Z]$ be an equation of C . Put $\mathbb{P}^2(\mathbb{F}_q) \setminus C(\mathbb{F}_q) = \{P_0, \dots, P_m\}$. For each i with $1 \leq i \leq m$, choose a linear form L_i over \mathbb{F}_q such that the line $\{L_i = 0\}$ passes through P_i but does not P_0 . Then $\{L_1 \cdots L_m F = 0\}(\mathbb{F}_q) = \mathbb{P}^2(\mathbb{F}_q) \setminus \{P_0\}$. Hence $\deg L_1 \cdots L_m F \geq 2q - 1$ by Corollary 2.4. Therefore $m + 1 \geq 2q - d$.

Next we will show that P_0, \dots, P_m are collinear if $m = 2q - d - 1$. For two indexes i, j with $1 \leq i < j \leq m$, if the line $\overline{P_i P_j}$ joining P_i and P_j does not contain P_0 , then the curve

$$D := \overline{P_i P_j} \cup \left\{ \prod_{\substack{k \text{ with} \\ k \neq i, j}} L_k \cdot F = 0 \right\}$$

is of degree $m - 1 + d = 2q - 2$ and $D(\mathbb{F}_q) = \mathbb{P}^2(\mathbb{F}_q) \setminus \{P_0\}$, which contradicts to Corollary 2.4. Hence P_0, \dots, P_m are collinear. \square

Theorem 3.3 *Fix an integer d with $q + 2 \leq d \leq 2q - 1$. Let C be a plane curve over \mathbb{F}_q of degree d such that $N_q(C) = q^2 + (d - q + 1)$. Then C is projectively equivalent to the curve defined by*

$$(X^q - XZ^{q-1})f(X, Y, Z) + (Y^q - YZ^{q-1})g(X, Y, Z) = 0$$

over \mathbb{F}_q , where $f(X, Y, Z)$ and $g(X, Y, Z)$ are of degree $d - q$ and $Xf(X, Y, 0) + Yg(X, Y, 0) = 0$ has $d - q + 1$ distinct roots on $\mathbb{P}^1(\mathbb{F}_q)$ with homogeneous coordinates X, Y .

Conversely such a curve has $q^2 + (d - q + 1)$ \mathbb{F}_q -points. (But we don't know whether the curve has an \mathbb{F}_q -line component or not.) Moreover if $q > 3$, then there is a curve without \mathbb{F}_q -line components among those curves.

Proof. From Lemma 3.2, the missing $2q - d$ points are collinear. Choosing coordinates X, Y, Z of \mathbb{P}^2 as those points lie on the line $Z = 0$. Hence the equation of C is in the ideal $(X^q - XZ^{q-1}, Y^q - YZ^{q-1})$ by Corollary 2.5. So C is defined by

$$(X^q - XZ^{q-1})f(X, Y, Z) + (Y^q - YZ^{q-1})g(X, Y, Z) = 0,$$

where $f(X, Y, Z), g(X, Y, Z) \in \mathbb{F}_q[X, Y, Z]$ are homogeneous of degree $d - q$. Since $C(\mathbb{F}_q) \cap \{Z = 0\}$ consists of $d - q + 1$ points,

$$X^q f(X, Y, 0) + Y^q g(X, Y, 0) = 0 \quad (4)$$

has $d - q + 1$ roots on $\mathbb{P}^2(\mathbb{F}_q)$ with coordinates X, Y . Since we consider only \mathbb{F}_q -points, (4) is equivalent to

$$Xf(X, Y, 0) + Yg(X, Y, 0) = 0. \quad (5)$$

Conversely it is obvious that such a curve has $q^2 + (d - q + 1)$ \mathbb{F}_q -points.

Now we construct a required equation if $q > 3$. Since $d \leq 2q - 1$, we can choose $d - q + 1$ distinct elements $\alpha_1, \dots, \alpha_{d-q+1}$ in \mathbb{F}_q . Let $\beta_0 (= 1), \beta_1, \dots, \beta_{d-q+1}$ be elements of \mathbb{F}_q determined by

$$\prod_{i=1}^{d-q+1} (Y - \alpha_i X) = \sum_{i=0}^{d-q+1} \beta_{d-q+1-i} X^{d-q+1-i} Y^i.$$

For $\mathbf{c} = (c_1, \dots, c_{d-q}) \in \mathbb{F}_q^{d-q}$, we consider the polynomial

$$\begin{aligned} F_{\mathbf{c}}(X, Y, Z) = & (X^q - XZ^{q-1}) \left(\sum_{i=0}^{d-q} \beta_{d-q+1-i} X^{d-q-i} Y^i + \sum_{i=1}^{d-q} c_i X^{d-q-i} Z^i \right) \\ & + (Y^q - YZ^{q-1}) Y^{d-q}. \end{aligned} \quad (6)$$

Let C be the curve $F_{\mathbf{c}}(X, Y, Z) = 0$. The claims are

Claim 1. $N_q(C) = q^2 + (d - q + 1)$;

Claim 2. C has no \mathbb{F}_q -line components if one choose $\mathbf{c} \in \mathbb{F}_q^{d-q}$ appropriately.

For Claim 1, it is obvious $C(\mathbb{F}_q) \supset \mathbb{A}^2(\mathbb{F}_q)$, where $\mathbb{A}^2 = \mathbb{P}^2 \setminus \{Z = 0\}$. From (5), $C(\mathbb{F}_q) \cap \{Z = 0\}$ comes from

$$\begin{aligned} 0 &= \sum_{i=0}^{d-q} \beta_{d-q+1-i} X^{d-q+1-i} Y^i + Y^{d-q+1} \\ &= \prod_{i=1}^{d-q+1} (Y - \alpha_i X). \end{aligned}$$

Hence $C(\mathbb{F}_q) \cap \{Z = 0\} = \{(1, \alpha_i, 0) \mid i = 1, 2, \dots, d - q + 1\}$, and $N_q(C) = q^2 + (d - q + 1)$.

We see the second claim. An \mathbb{F}_q -line which is a component of C must pass through one of the $d - q + 1$ \mathbb{F}_q -points on $Z = 0$ above, and can't be the line $Z = 0$. So the possible \mathbb{F}_q -lines that may be components of C are

$$\{Y - \alpha X = \rho Z \mid \alpha \in \{\alpha_1, \dots, \alpha_{d-q+1}\}, \rho \in \mathbb{F}_q\}.$$

By direct computation, $F_{\mathbf{c}}(X, \alpha X + \rho Z, Z)$ can be written as

$$(X^q - XZ^{q-1}) \left(\sum_{i=0}^{d-q} \gamma_i(\alpha, \rho) X^{d-q-i} Z^i + \sum_{i=1}^{d-q} c_i X^{d-q-i} Z^i \right).$$

The number of vectors in

$$\{(\gamma_0(\alpha, \rho), \dots, \gamma_{d-q}(\alpha, \rho)) \mid \alpha \in \{\alpha_1, \dots, \alpha_{d-q+1}\}, \rho \in \mathbb{F}_q\}$$

is at most $(d - q + 1)q$ and that of the vectors $\{(0, \mathbf{c}) \mid \mathbf{c} \in \mathbb{F}_q^{d-q}\}$ is q^{d-q} . Since $(d - q + 1)q < q^{d-q}$ if $q + 2 \leq d \leq 2q - 1$ and $q > 3$, we can choose a vector \mathbf{c} such that $F_{\mathbf{c}}(X, \alpha X + \rho Z, Z)$ is nontrivial for all pairs (α, ρ) . \square

Remark 3.4 Suppose that $q > 3$ and $d \geq 2q$. Let $\{e_\alpha \mid \alpha \in \mathbb{F}_q\}$ be integers such that $e_\alpha \geq 1$ for any $\alpha \in \mathbb{F}_q$ and $\sum_{\alpha \in \mathbb{F}_q} e_\alpha = d - q + 1$. Let $\beta_0, \dots, \beta_{d-q+1}$ be elements of \mathbb{F}_q determined by

$$\prod_{\alpha \in \mathbb{F}_q} (Y - \alpha X)^{e_\alpha} = \sum_{i=0}^{d-q+1} \beta_{d-q+1-i} X^{d-q+1-i} Y^i$$

as in the proof of Theorem 3.3. Then, by just imitating the proof of the claims, we can find $(c_1, \dots, c_{d-q}) \in \mathbb{F}_q^{d-q}$ such that the curve C with the equation $F_{\mathbf{c}}(X, Y, Z) = 0$ defined by (6) has the property

- (i) $N_q(C) = \theta_q(2) - 1$;
- (ii) no \mathbb{F}_q -line is a component of C .

Proof of Theorem 1.2 (1). (i) and (ii) are consequences of Theorem 3.3 and Remark 3.4. If $d = q + 2$, any curve C over \mathbb{F}_q without \mathbb{F}_q -line components having $q^2 + 3$ \mathbb{F}_q -points is absolutely irreducible by Lemma 3.1. \square

3.2 The case $d = q + 1$

Theorem 3.5 *Let \mathbb{A}^2 be the affine part of \mathbb{P}^2 defined by $Z \neq 0$, and C a curve of degree d in \mathbb{P}^2 defined over \mathbb{F}_q .*

- (i) *If $C(\mathbb{F}_q) = \mathbb{A}^2(\mathbb{F}_q)$, then $d \geq q + 1$.*
- (ii) *When $d = q + 1$, $C(\mathbb{F}_q) = \mathbb{A}^2(\mathbb{F}_q)$ if and only if C is defined by an equation of the following type:*

$$(X^q - XZ^{q-1}, Y^q - YZ^{q-1}) \begin{pmatrix} a_0 & a_1 & a_2 \\ b_0 & b_1 & b_2 \end{pmatrix} \begin{pmatrix} X \\ Y \\ Z \end{pmatrix} = 0, \quad (7)$$

where $a_0, \dots, b_2 \in \mathbb{F}_q$ with the polynomial

$$(s, t) \begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix} \begin{pmatrix} s \\ t \end{pmatrix} \quad (8)$$

in s and t being irreducible over \mathbb{F}_q .

- (iii) *The curve described above is absolutely irreducible with a unique singular points, which is an \mathbb{F}_q -points.*

Proof. (i) Since $C(\mathbb{F}_q) = \mathbb{A}^2(\mathbb{F}_q)$, the curve has no \mathbb{F}_q -linear component. Hence we can apply the Sziklai bound to C , namely, $q^2 \leq (d - 1)q + 1$. So $d \geq q + 1$.

(ii) From Corollary 2.5, the equation of C can be written as (7). Since the line $Z = 0$ does not meet with $C(\mathbb{F}_q)$, the polynomial (8) must be irreducible over \mathbb{F}_q , and vice versa.

(iii) Since $\deg C = q + 1$ and $N_q(C) = q^2$, C is absolutely irreducible if $q \geq 3$ by Lemma 3.1. Suppose $q = 2$. Then $\deg C = 3$. Further suppose that C is reducible. Since C can't have \mathbb{F}_q -line components, C must be a union of three conjugate \mathbb{F}_{q^3} -lines over \mathbb{F}_q . Hence $N_q(C) \leq 1$, which is a contradiction. Hence C is absolutely irreducible even if $q = 2$.

Since (8) is irreducible over \mathbb{F}_q , $\det \begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix} \neq 0$. Let (x_0, y_0) be the unique solution of

$$\begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = - \begin{pmatrix} a_2 \\ b_2 \end{pmatrix}.$$

Since $(x_0, y_0, 1) \in \mathbb{A}^2(\mathbb{F}_q)$, it is a point of C . Choose new coordinates X', Y', Z' as

$$\begin{cases} X' &= X - x_0 Z \\ Y' &= Y - y_0 Z \\ Z' &= Z \end{cases}.$$

Then the point $(x_0, y_0, 1)$ goes to $(0, 0, 1)$ in these new coordinates. Let $x = X'/Z'$ and $y = Y'/Z'$ be affine coordinates of $\mathbb{P}^2 \setminus \{Z' = 0\}$ with respect to these new coordinates, and $O = (0, 0, 1)$. The equation of C in x, y is $f(x, y) = 0$, where

$$\begin{aligned} f(x, y) &= (x^q - x)(a_0x + a_1y) + (y^q - y)(b_0x + b_1y) \\ &= x^q(a_0x + a_1y) + y^q(b_0x + b_1y) - (a_0x^2 + (a_1 + b_0)xy + b_1y^2). \end{aligned} \quad (9)$$

Note that the condition of irreducibility of (8) in the original coordinates corresponds to that of the quadric terms in (9) in these affine coordinates. Hence O is an ordinary double point. In fact, the embedded tangent cone at O is defined by the quadric terms in (9) which is irreducible over \mathbb{F}_q , and then it splits into two \mathbb{F}_q -conjugate lines over \mathbb{F}_{q^2} .

Let us investigate the intersection of C and any line l passing through O over the algebraic closure of \mathbb{F}_q . If $l = \{x = 0\}$, then the y -coordinates of points of $C \cap l$ is given by $b_1(y^{q+1} - y^2)$. Since $b_1 \neq 0$ (otherwise the quadric terms in (9) is reducible over \mathbb{F}_q), each point of $C \cap l \setminus \{O\}$ is nonsingular. Next we choose a line l_c defined by $y = cx$. Substitute cx for y in $f(x, y)$. Then

$$((a_0 + ca_1) + c^q(b_0 + cb_1))x^{q+1} - ((a_0 + ca_1) + c(b_0 + cb_1))x^2 = 0. \quad (10)$$

If both coefficients of x^{q+1} and x^2 are nonzero, then each point of $C \cap l_c \setminus \{O\}$ is nonsingular.

Suppose that the coefficient of x^2 is 0, namely $(a_0 + ca_1) + c(b_0 + cb_1) = 0$. Then $(1, c)$ is a root of $a_0x^2 + (a_1 + b_0)xy + b_1y^2 = 0$, and vice versa. Note that this quadric is the lowest terms of the local equation (9) of C around O . Hence $(a_0 + ca_1) + c(b_0 + cb_1) = 0$ if and only if l_c is the embedded tangent line of one of the two branches of the singular points O . In this case, $c \notin \mathbb{F}_q$ because the quadric is irreducible over \mathbb{F}_q . So $(a_0 + ca_1) + c^q(b_0 + cb_1) \neq 0$. (Actually, $(a_0 + ca_1) + c^q(b_0 + cb_1) = 0$ and $(a_0 + ca_1) + c(b_0 + cb_1) = 0$ imply $(c^q - c)(b_0 + cb_1) = 0$. Since b_1 and b_0 are in \mathbb{F}_q and at least one of them is nonzero, $c \in \mathbb{F}_q$, which is a contradiction.) Hence $C \cap l_c = \{O\}$.

Finally suppose that

$$(a_0 + ca_1) + c^q(b_0 + cb_1) = 0. \quad (11)$$

Then l_c meets with $C \setminus \{O\}$ at $(1, c, 0)$ of multiplicity $q - 1$. Choose local coordinates around $(1, c, 0)$ as $t := \frac{Y'}{X'} - c$ and $z := \frac{Z'}{X'}$. Then the local equation of C is

$$\begin{aligned} 0 &= (1 - z^{q-1})(a_0 + a_1(c + t)) + (c^q + t^q - (c + t)z^q)(b_0 + b_1(c + t)) \\ &= (a_1 + b_1c^q)t + (\text{higher terms in } t \text{ and } z). \end{aligned}$$

Here, if $a_1 + b_1c^q = 0$, then $a_0 + b_0c^q = 0$ because the assumption (11). This is a contradiction because $\det \begin{pmatrix} a_0 & a_1 \\ b_0 & b_1 \end{pmatrix} \neq 0$. Hence the coefficient of t in the local equation is nonzero, and hence C is nonsingular at $(1, c, 0)$. \square

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